The microstructural foundations of rough volatility

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Main classes of volatility models

Prices are often modeled as continuous semi-martingales of the form

$$dP_t = P_t(\mu_t dt + \sigma_t dW_t).$$

The volatility process σ_s is the most important ingredient of the model. The three most classical classes of volatility models are :

- Deterministic volatility (Black and Scholes 1973),
- Local volatility (Dupire 1994, Derman and Kani 1994),
- Stochastic volatility (Hull and White 1987, Heston 1993, Hagan *et al.* 2002,...).

However, it has been recently shown that models where the volatility is driven by a fractional Brownian motion (and not a classical Brownian motion) enable us to reproduce very well the behavior of historical data and of the volatility surface.

Fractional Brownian motion (I)

Definition

The fractional Brownian motion (fBm) with Hurst parameter H is the only process W^H to satisfy :

- Self-similarity : $(W^H_{at}) \stackrel{\mathcal{L}}{=} a^H (W^H_t)$.
- Stationary increments : $(W^H_{t+h} - W^H_t) \stackrel{\mathcal{L}}{=} (W^H_h)$.
- Gaussian process with $\mathbb{E}[W^H_1] = 0$ and $\mathbb{E}[(W^H_1)^2] = 1$.

Fractional Brownian motion (II)

Proposition

For all $\varepsilon > 0$, W^H is $(H-\varepsilon)$-Hölder a.s.

Proposition

The absolute moments of the increments of the fBm satisfy

$$\mathbb{E}[|W^H_{t+h} - W^H_t|^q] = K_q h^{Hq}.$$

Proposition

If $H > 1/2$, the fBm exhibits long memory in the sense that

$$\mathrm{Cov}[W^H_{t+1} - W^H_t, W^H_1] \sim \frac{C}{t^{2-2H}}.$$

Fractional models

FSV model

Some models have been built using fractional Brownian motion with Hurst parameter $H > 1/2$ to reproduce the supposed long memory property of the volatility :

- Comte and Renault 1998 (FSV model) :

$$d\log(\sigma_t) = \nu dW_t^H + \alpha(m - \log(\sigma_t))dt.$$

Here α is large to model a mean reversion effect.

Fractional models

RFSV model

However, statistical investigation of recent prices and options data rather suggests the use of rough versions of the preceding model, for example :

$$d\log(\sigma_t) = \nu dW_t^H + \alpha(m - \log(\sigma_t))dt,$$

with H of order 0.1 and α very small (Rough FSV model).

Example

Volatility of the S&P

- Everyday, we estimate the volatility of the S&P at 11am (say), over 3500 days.
- We study the quantity

$$m(\Delta, q) = \mathbb{E}[|\log(\sigma_{t+\Delta}) - \log(\sigma_t)|^q],$$

for various q and Δ, the smallest Δ being one day.

- In the RFSV model $m(\Delta, q) \sim c\Delta^{qH}$.

The log-volatility

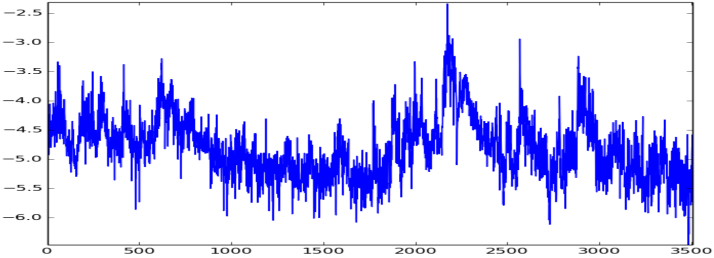

FIGURE : The log volatility $\log(\sigma_t)$ as a function of t, S&P.

Example : Scaling of the moments

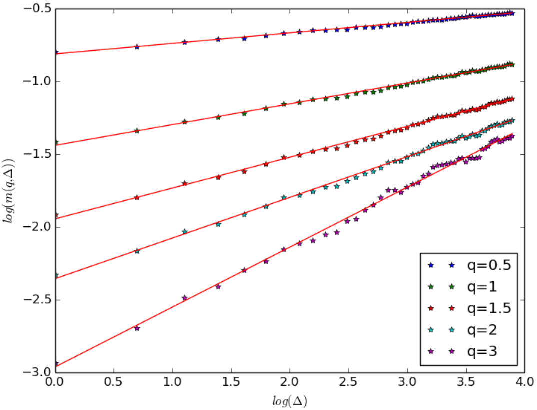

FIGURE : $\log(m(q,\Delta)) = \zeta_q \log(\Delta) + C_q$. The scaling is not only valid as Δ tends to zero, but holds on a wide range of time scales.

Example : Monofractality of the log-volatility

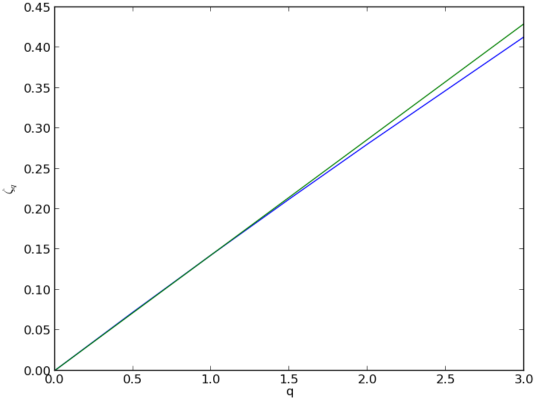

FIGURE : Empirical ζ_q and $q \to Hq$ with $H = 0.14$ (similar to a fBm with Hurst parameter H).

Properties of RFSV-type models

Statistical analysis of the RFSV model

- Reproduces very well (almost) all the statistical stylized facts of volatility, with explicit formulas.
- Very good fit of the volatility surface, in particular of the ATM skew.
- No power law long memory property.
- Applied to the RFSV model, statistical tests for long memory behave the same way as for real data and deduce, probably wrongly, the presence of long memory in the volatility.
- Explicit prediction formulas for the future volatility, depending only on the parameter H, outperforming classical predictors. To forecast the volatility at time $t + \Delta$, one needs to consider the data in the past until $t - \Delta$.

Multiscaling in finance

- An important property of volatility time series is their multiscaling behavior, see Mantegna and Stanley 2000 and Bouchaud and Potters 2003.
- This means one observes essentially the same law whatever the time scale.
- In particular, there are periods of high and low market activity at different time scales.
- Very few models reproduce this property, see multifractal models.

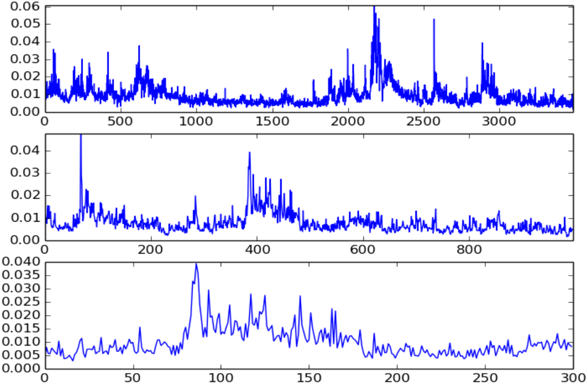

FIGURE : Empirical volatility over 10, 3 and 1 years.

Our model on different time intervals

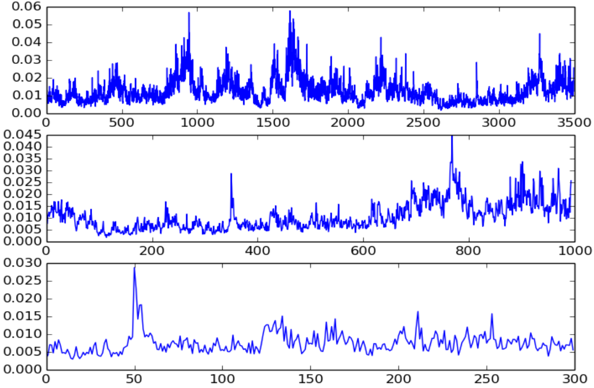

FIGURE : Simulated volatility over 10, 3 and 1 years. We observe the same alternations of periods of high market activity with periods of low market activity.

Apparent multiscaling in our model

- Let $L^{H,\nu}$ be the law on $[0,1]$ of the process $e^{\nu W_t^H}$.
- Then the law of the volatility process on $[0,T]$ renormalized on $[0,1]$: σ_{tT}/σ_0 is $L^{H,\nu T^H}$.
- If one observes the volatility on $T = 10$ years (2500 days) instead of $T = 1$ day, the parameter νT^H defining the law of the volatility is only multiplied by $2500^H \sim 3$.
- Therefore, one observes quite the same properties on a very wide range of time scales.
- The roughness of the volatility process ($H = 0.14$) implies a multiscaling behavior of the volatility.

Leverage effect and rough volatility

Leverage effect

- The leverage effect is a well studied phenomenon : negative correlation between price increments and volatility increments.
- Very easy to incorporate within a rough volatility framework : Use Mandelbrot-van Ness representation of the fractional Brownian motion :

$$W_t^H = \int_0^t \frac{dW_s}{(t-s)^{\frac{1}{2}-H}} + \int_{-\infty}^0 \left(\frac{1}{(t-s)^{\frac{1}{2}-H}} - \frac{1}{(-s)^{\frac{1}{2}-H}}\right) dW_s,$$

and correlate W with the Brownian motion driving the price.

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Building the model

Necessary conditions for a good microscopic price model

We want :

- A tick-by-tick model.
- A model reproducing the stylized facts of modern electronic markets in the context of high frequency trading.
- A model helping us to understand the rough dynamics of the volatility from the high frequency behaviour of market participants.
- A model helping us to understand leverage effect.

Building the model

Stylized facts 1-2

- Markets are highly endogenous, meaning that most of the orders have no real economic motivations but are rather sent by algorithms in reaction to other orders, see Bouchaud *et al.*, Filimonov and Sornette.
- Mechanisms preventing statistical arbitrages take place on high frequency markets, meaning that at the high frequency scale, building strategies that are on average profitable is hardly possible.

Building the model

Stylized facts 3-4

- There is some asymmetry in the liquidity on the bid and ask sides of the order book. In particular, a market maker is likely to raise the price by less following a buy order than to lower the price following the same size sell order, see Brennan *et al.*, Brunnermeier and Pedersen, Hendershott and Seasholes.
- A large proportion of transactions is due to large orders, called metaorders, which are not executed at once but split in time.

Building the model

Hawkes processes

- Our tick-by-tick price model is based on Hawkes processes in dimension two, very much inspired by the approaches in Bacry *et al.* and Jaisson and R.
- A two-dimensional Hawkes process is a bivariate point process $(N_t^+, N_t^-)_{t\geq 0}$ taking values in $(\mathbb{R}^+)^2$ and with intensity $(\lambda_t^+, \lambda_t^-)$ of the form :

$$\begin{pmatrix}\lambda_t^+\\ \lambda_t^-\end{pmatrix} = \begin{pmatrix}\mu^+\\ \mu^-\end{pmatrix} + \int_0^t \begin{pmatrix}\varphi_1(t-s) & \varphi_3(t-s)\\ \varphi_2(t-s) & \varphi_4(t-s)\end{pmatrix} . \begin{pmatrix}dN_s^+\\ dN_s^-\end{pmatrix}.$$

Building the model

The microscopic price model

- Our model is simply given by

$$P_t = N_t^+ - N_t^-.$$

- N_t^+ corresponds to the number of upward jumps of the asset in the time interval $[0, t]$ and N_t^- to the number of downward jumps. Hence, the instantaneous probability to get an upward (downward) jump depends on the location in time of the past upward and downward jumps.
- By construction, the price process lives on a discrete grid.
- Statistical properties of this model have been studied in details.

Encoding the stylized facts

The right parametrization of the model

- Recall that

$$\begin{pmatrix} \lambda_t^+ \\ \lambda_t^- \end{pmatrix} = \begin{pmatrix} \mu^+ \\ \mu^- \end{pmatrix} + \int_0^t \begin{pmatrix} \varphi_1(t-s) & \varphi_3(t-s) \\ \varphi_2(t-s) & \varphi_4(t-s) \end{pmatrix} \cdot \begin{pmatrix} dN_s^+ \\ dN_s^- \end{pmatrix}.$$

- High degree of endogeneity of the market$\rightarrow$ L^1 norm of the largest eigenvalue of the kernel matrix close to one.
- No arbitrage$\rightarrow$ $\varphi_1 + \varphi_3 = \varphi_2 + \varphi_4$.
- Liquidity asymmetry$\rightarrow$ $\varphi_3 = \beta\varphi_2$, with $\beta > 1$.
- Metaorders splitting$\rightarrow$ $\varphi_1(x),\ \varphi_2(x) \underset{x\rightarrow\infty}{\sim} K/x^{1+\alpha}$, $\alpha \approx 0.6$.

About the degree of endogeneity of the market

L^1 norm close to unity

- For simplicity, let us consider the case of a Hawkes process in dimension 1 with Poisson rate μ and kernel ϕ :

$$\lambda_t = \mu + \int_{(0,t)} \phi(t-s)dN_s.$$

- N_t then represents the number of transactions between time 0 and time t.
- L^1 norm of the largest eigenvalue close to unity$\rightarrow$$L^1$ norm of ϕ close to unity. This is systematically observed in practice, see Hardiman, Bercot and Bouchaud ; Filimonov and Sornette.
- The parameter $\|\phi\|_1$ corresponds to the so-called degree of endogeneity of the market.

About the degree of endogeneity of the market

Population interpretation of Hawkes processes

- Under the assumption $\|\phi\|_1 < 1$, Hawkes processes can be represented as a population process where migrants arrive according to a Poisson process with parameter μ.
- Then each migrant gives birth to children according to a non homogeneous Poisson process with intensity function ϕ, these children also giving birth to children according to the same non homogeneous Poisson process, see Hawkes (74).
- Now consider for example the classical case of buy (or sell) market orders. Then migrants can be seen as exogenous orders whereas children are viewed as orders triggered by other orders.

About the degree of endogeneity of the market

Degree of endogeneity of the market

- The parameter $\|\phi\|_1$ corresponds to the average number of children of an individual, $\|\phi\|_1^2$ to the average number of grandchildren of an individual,... Therefore, if we call cluster the descendants of a migrant, then the average size of a cluster is given by $\sum_{k\geq 1}\|\phi\|_1^k = \|\phi\|_1/(1-\|\phi\|_1)$.
- Thus, the average proportion of endogenously triggered events is $\|\phi\|_1/(1-\|\phi\|_1)$ divided by $1+\|\phi\|_1/(1-\|\phi\|_1)$, which is equal to $\|\phi\|_1$.

The scaling limit of the price model

Limit theorem

After suitable scaling in time and space, the long term limit of our price model satisfies the following Rough Heston dynamics :

$$P_t = \int_0^t \sqrt{V_s} dW_s - \frac{1}{2} \int_0^t V_s ds,$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \lambda(\theta - V_s) ds + \frac{\lambda\nu}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sqrt{V_s} dB_s,$$

with

$$d\langle W, B \rangle_t = \frac{1-\beta}{\sqrt{2(1+\beta^2)}} dt.$$

The scaling limit of the price model

Comments on the theorem

- The Hurst parameter $H = \alpha - 1/2$.
- Hence stylized facts of modern market microstructure naturally give rise to fractional dynamics and leverage effect.
- One of the only cases of scaling limit of a non ad hoc "micro model" where leverage effect appears in the limit. Compare with Nelson's limit of GARCH models for example.

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Using rough volatility models

Pricing and hedging with rough volatility

- Non Markovian, non semi-martingale volatility process$\rightarrow$ standard approaches cannot be applied.
- Monte-Carlo methods are hard to use.
- No hope for explicit formulas, except in one particular case : Rough Heston model.
- Approximations are needed$\rightarrow$ Josselin's talk.

Rough Heston model

Rough version of Heston model

- We consider the following model :

$$dS_t = S_t\sqrt{V_t}dW_t,$$

$$V_t = V_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\lambda(\theta - V_s)ds + \frac{\lambda\nu}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\sqrt{V_s}dB_s,$$

with

$$d\langle W, B\rangle_t = \rho dt.$$

Computing the characteristic functions

From microstructure to option prices

- Deriving characteristic functions for our microscopic Hawkes-based price model and passing to the limit, we are able to compute characteristic functions in the Rough Heston model.

Characteristic functions

We write :

$$I^{1-\alpha}f(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^x \frac{f(t)}{(x-t)^\alpha}dt, \;\; D^\alpha f(x) = \frac{d}{dx}I^{1-\alpha}f(x).$$

Theorem

The characteristic function at time t for the Rough Heston model is given by

$$L(a,t) = \exp\Big(\int_0^t g(a,s)ds + \frac{V_0}{\theta\lambda}I^{1-\alpha}g(a,t)\Big),$$

with $g(a,)$ the unique solution of the fractional Riccati equation :

$$D^\alpha g(a,s) = \frac{\lambda\theta}{2}(-a^2 - ia) + \lambda(ia\rho\nu - 1)g(a,s) + \frac{\lambda\nu^2}{2\theta}g^2(a,s).$$